MATH226

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Contents

9.1 Three-Dimensional Coordinate Systems

Theorem 9.1.1: Distance Formula in Three Dimensions

The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Theorem 9.1.2: Equation of a Square

An equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x - h)^2 + (y - h)^2 + (z - l)^2 = r^2$$

In particular, if the center is the origin O, then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

9.2 Vectors

Definition 9.2.1: Vector Addition

If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum of u + v** is the vector from the initial point of **u** to the terminal point of **v**.

Definition 9.2.2: Scalar Multiplication

If c is a scalar and **v**is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of **v** and whose direction is the same as **v** if $c > 0$ and is opposite to v if $c < 0$. If $c = 0$ or $v = 0$, then $c\mathbf{v} = \mathbf{0}$.

Corollary 9.2.3

The length of the two-dimensional vector $\mathbf{a} = <a_1, a_2>$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = <a_1, a_2, a_3>$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

9.3 The Dot Product

Definition 9.3.1: Dot Product

If $\mathbf{a} = <s_1, a_2, a_3>$ and $\mathbf{b}<b_1, b_2, b_3>$, then the **dot product** of **a** and **b** is the number $\boldsymbol{a} \cdot \boldsymbol{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Theorem 9.3.2

If θ is the angle between the vectors **a** and **b**, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

Corollary 9.3.3

If θ is the angle between the nonzero vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Theorem 9.3.4: Orthogonal

Two vectors **a** and **b** are orthogonal if and only if $\boldsymbol{a} \cdot \boldsymbol{b}$=0.

$$\mathbf{u} \perp \mathbf{v} \Leftrightarrow \|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Definition 9.3.5: Projections

Scalar projection (Component) of **b** onto **a**:

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

Vector projections of **b** onto **a**:

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \cdot \mathbf{a}$$

9.4 The Cross Product

Definition 9.4.1: Cross Product

If $\mathbf{a} = <a_1, a_2, a_3>$ and $\mathbf{b} = <b_1, b_2, b_3>$, then the **cross product** of **a** and **b** is the vector

$$\mathbf{a} \times \mathbf{b} =< a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 >$$

Theorem 9.4.2

The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Theorem 9.4.3

If θ is the angle between $\mathbf{a}$ ans $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

Corollary 9.4.4: Parallel

Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = 0$$

$$\mathbf{a} \times \mathbf{b} = 0 \quad \Leftrightarrow \quad \mathbf{a} \parallel \mathbf{b} \quad \Leftrightarrow \quad \mathbf{b} = k\mathbf{a} \quad \text{for k} \in \mathbb{R}$$

Definition 9.4.5: Scalar Triple Product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Property 9.4.6: Area and Volumes

Area of parallelogram:

$$A_{\text{parallelogram}} = \|\mathbf{a} \times \mathbf{b}\|$$

Volume of parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

If the volume of the parallelepiped determined by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is 0, then the vectors must lie in the same plane; that is , they are **coplanar**,

Property 9.4.7: Directing and Normal Vector

If a line in $\mathbb{R}^2$ is directed by $\mathbf{u} < \alpha, \beta >$, then $\mathbf{n} < -\beta, \alpha >$ is a vector $\perp$ to this line.

$$\mathbf{u} \cdot \mathbf{n} = 0 \Leftrightarrow \mathbf{u} \perp \mathbf{n}$$

9.5 Equations of Lines and Planes

Definition 9.5.1: Vector Equation

The **vector equation** of L. Each value of of the **parameter** t gives the position vector $\mathbf{r}$ of a point on L.

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$$

Definition 9.5.2: Parametric Equation of *L*

Parametric equation of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} =< a, b, c >$. Each value of the parameter t gives a point (x, y, z) on L.

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

The numbers a, b, c are called **direction numbers** of L.

Definition 9.5.3: Symmetric Equations of *L*

The following equation is called **Symmetric Equations** of L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} =< a, b, c >$.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Definition 9.5.4: Line Segment

The line segment from $\mathbf{r_0}$ to $\mathbf{r_1}$ is given by the vector equation

$$\mathbf{r} = (1 - t)\mathbf{r_0} + t\mathbf{r_1}$$

Definition 9.5.5: Vector Equation of the Plane

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector that is orthogonal to the plane. This orthogonal vector $\mathbf{n}$ is called a **normal vector**.

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$$

which can be rewritten as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$$

Definition 9.5.6: Scalar Equation of Plane

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is the **Scalar Equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $n = < a, b, c >$**

Definition 9.5.7: Linear Equation of Plane

$$ax + by + cz + d = 0$$

where $d = -(ax_0 + by_0 + cz_0)$. This is called a **linear equation** in x, y, z.
For such an equation, **normal vector** is $< a, b, c >$.
Two planes are **parallel** if their normal vectors are parallel.

Theorem 9.5.8: Distance from Point to Plane

The distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

9.6 Cylinders and Quadric Surfaces

Definition 9.6.1: Cylinder

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a give plane curve.

Parabolic Cylinder:

$$z = x^2$$

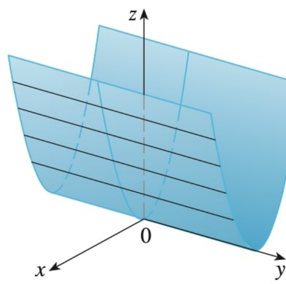
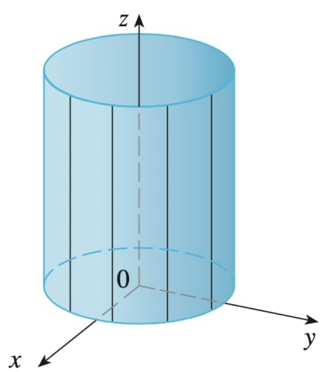


Figure 9.1

Circular Cylinder:

$$x^2 + y^2 = 1$$



Definition 9.6.2: Quadric Surfaces

A **quadric surface** is the graph of a second-degree equation in three variables x, y, and z. It can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad Ax^2 + By^2 + Iz = 0$$

Property 9.6.3: Graph of Quadric Surfaces

Ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Cone:

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Elliptic Paraboloid:

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Hyperbolic Paraboloid:

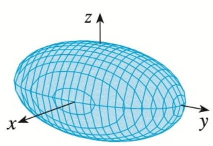
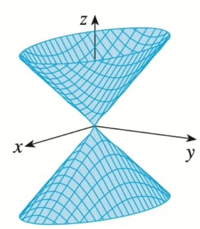
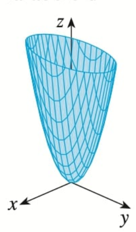
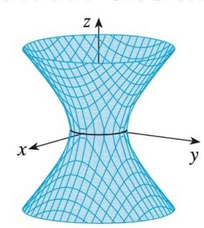
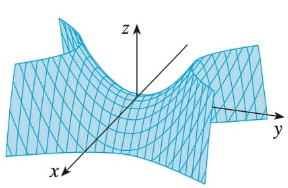
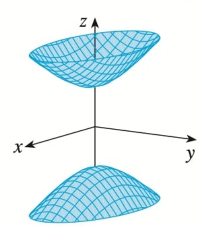
$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Hyperboloid of One Sheet:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Hyperboloid of Two Sheets:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

9.7 Vector Functions and Space Curves

A **vector-valued function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors. If $f(t), g(t), h(t)$ are components of the vector $\mathbf{r}(t)$, then f, g, h are real-valued functions called **component functions** of **r** and we can write

$$\mathbf{r}(t) =< f(t), g(t), h(t) >= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Theorem 9.7.1: Limit of Vector Function

If $\mathbf{r}(t) =< f(t), g(t), h(t) >$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

A function **r** is **continuous at** a if

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$$

Definition 9.7.2: Space Curve

The set C of all points (x, y, z) in space, where

$$x = f(t) \quad y = g(t) \quad z = h(t)$$

and t varies throughout the interval I, is called a **space curve**. The equations are called **parametric equations of C** and t is called a **parameter**.

Definition 9.7.3: Derivative r'

The **derivative r'** of a vector function **f** is defined:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}' = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

Theorem 9.7.4

If $\mathbf{r}(t) =< f(t), g(t), h(t) >= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, h are differentiable functions, then

$$\mathbf{r}'(t) =< f'(t), g'(t), h'(t) >= f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Theorem 9.7.5: Differentiation Rules

Suppose **u** and **v** are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

(1)

$$\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$$

(2)

$$\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

(3)

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

(4)

$$\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)'] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$$

(5)

$$\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$$

(6) Chain Rule

$$\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$$

Theorem 9.7.6: Integrals

The **definite integral** of a continuous vector function $\mathbf{r}(t)$ can be defined

$$\int_a^b \mathbf{r}(t)\ \mathrm{d}t = \Big(\int_a^b f(t)\ \mathrm{d}t\Big)\mathbf{i} + \Big(\int_a^b g(t)\ \mathrm{d}t\Big)\mathbf{j} + \Big(\int_a^b h(t)\ \mathrm{d}t\Big)\mathbf{k}$$

9.8 Arc Length and Curvature

Theorem 9.8.1: Arc Length

If the curve is traversed exactly once as t increases from a to b, then it can be shown that its length is

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\ \mathrm{d}t \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ \mathrm{d}t \\ &= \int_a^b |\mathbf{r}'(t)|\ \mathrm{d}t \end{aligned}$$

Definition 9.8.2: Arc Length Function

We say that two different equations of the same curve are **parametrizations** of the curve. We define **r**'s **arc length function** s by

$$s(t) = \int_a^t |\mathbf{r}'(u)|\ \mathrm{d}u$$

If we differentiate both sides of this theorem of the FTC, we obtain

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

It is often useful to **parametrize a curve with respect to arc length** because arc length arises naturally from the shape.

Definition 9.8.3: Tangent, Normal and Binormal Vectors

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \qquad \mathbf{B}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|} \qquad \mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t)$$

$$\mathbf{r}''(t) = \alpha(t)\mathbf{T}(t) + \beta(t)\mathbf{N}(t) \qquad \alpha(t) = \frac{(\mathbf{r}' \cdot \mathbf{r}'')(t)}{\|\mathbf{r}'(t)\|} \qquad \beta(t) = \frac{\|(\mathbf{r}' \times \mathbf{r}'')(t)\|}{\|\mathbf{r}'(t)\|} = \sqrt{\|\mathbf{r}''(t)\| - \alpha(t)^2}$$

Definition 9.8.4: Curvature

The **curvature** of a curve is

$$\begin{aligned}\kappa &= \left|\frac{d\mathbf{T}}{ds}\right| \\ &= \frac{\|(\mathbf{r}' \times \mathbf{r}'')(t)\|}{\|\mathbf{r}'(t)\|^3} \\ &= \left\|\frac{d^2 r}{ds^2}\right\|\end{aligned}$$

If it is in a 2D plane, than it could be written as

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{\frac{3}{2}}}$$

Property 9.8.5: Geometric Properties of Parametric Curve Reparametrized by the Arc Length

(1)

$$\frac{d\mathbf{r}}{ds} = \mathbf{T}(s)$$

(2)

$$\frac{d\mathbf{T}}{ds} = \kappa(s)\mathbf{N}(s) = \frac{d^2\mathbf{r}}{ds^2} \qquad \text{with} \qquad \kappa(s) = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left\|\frac{d^2\mathbf{r}}{ds^2}\right\|$$

(3)

$$\mathbf{r}(s_0 + \Delta_s) = \mathbf{r}(s_0) + \Delta s\mathbf{T}(s_0) + \frac{(\Delta s)^2}{2}\kappa(s_0)\mathbf{N}(s_0)$$

(4) **Osculating Plane** is $\perp \mathbf{B}(t)$ throughout the point $\mathbf{r}(t)$. In particular, $\mathbf{r}''(t)$ is parallel to the Osculating Plane at $\mathbf{r}(t)$. It is directed by $<\mathbf{T}(t), \mathbf{N}(t)>$.

9.9 Motion in Space Velocity and Acceleration

Definition 9.9.1: Velocity and Acceleration

Velocity vector $v(t)$ at time t:

$$\mathbf{v}(t) = \mathbf{r}'(t)$$

Acceleration:

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

Tangential Component:

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}$$

Normal Component:

$$a_N = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}$$

Chapter 10

Partial Derivatives

10.1 Functions of Several Variables

Definition 10.1.1: Domain and Range

A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) | (x, y) \in D\}$.

Definition 10.1.2: Graph of f

If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) in D.

Definition 10.1.3: Level Curves

The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

10.2 Limits and Continuity

Definition 10.2.1: Limits

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that
if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$

Theorem 10.2.2: Existence of Limits

If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Definition 10.2.3: Continuity

A function f of two variables is called **continuous at** (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

We say f is **continuous on** D is continuous at every point (a,b) in D.

Property 10.2.4

If f is defined on a subset D of $\mathbb{R}^2$, then $\lim_{x\to a} f(\mathbf{x}) = L$ means that for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \epsilon$$

10.3 Partial Derivatives

Definition 10.3.1: Partial Derivative

If f is function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Definition 10.3.2: Notations for Partial Derivatives

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_1 f = D_y f$$

Theorem 10.3.3: Rule for Finding Partial Derivative of $z = f(x,y)$

(1) To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.

(2) To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Theorem 10.3.4: Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

10.4 Tangent Planes and Linear Approximations

Definition 10.4.1: Tangent Plane

Suppose f has continuous partial derivatives. An equation of the **tangent plane** to the surface $z = f(x,y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Definition 10.4.2: Linear Approximation

The linear function whose graph is the tangent plane, namely

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

is called **linearization** of f at (a,b) and the approximation

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a,b).

Definition 10.4.3: Differentiable

If $z = f(x,y)$, then f is **differentiable** at (a,b) if Δz can be expressed in the form

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

Theorem 10.4.4

If the partial derivative f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

Definition 10.4.5: Total Differential

The **differential** dz, also called the **total differential**, is defined by

$$\begin{aligned} dz &= f_x(x,y)dx + f_y(x,y)dy \\ &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \end{aligned}$$

10.5 The Chain Rule

Theorem 10.5.1: The Chain Rule (General Version)

Suppose that u is a differentiable function of the n variables $x_1, x_2, ..., x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, ..., t_m$. Then u is a function of $t_1, t_2, ..., t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + ... + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, ..., m$.

Theorem 10.5.2: Implicit Function Theorem

The **Implicit Function Theorem (Principle)**, gives conditions under which this assumption is valid. The theorem guarantees the existence of a function $f : B(r_0, a) \to B(r_1, b) \subset \mathbb{R}^k$ such that

$$F(x, f(x)) = 0$$

By IFP, the following assumption holds true:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

$$z_x = -\frac{F_x}{F_z} \qquad z_y = -\frac{F_y}{F_z}$$

10.6 Directional Derivatives and the Gradient Vector

Definition 10.6.1: Directional Derivative

The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} =< a, b >$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h\to 0}\frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Theorem 10.6.2: Directional Derivative

If f is a differentiable function of x and y, then f has a directional derivative in the direction of nay unit vector $\mathbf{u} =< a, b >$ and

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b$$

Definition 10.6.3: Gradient

If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x,y) =< f_x(x,y), f_y(x,y) >= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

With this notation for the gradient vector, the expression for the directional derivative can rewritten as

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$$

Definition 10.6.4: Directional and Gradients of Functions of Three Variables

The **directional derivatives** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} =< a, b, c >$ is

$$D_{\mathbf{u}}f(x_0,y_0,z_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0,y_0,z_0)}{h}$$

if this limit exists.

$$D_{\mathbf{u}}f(x_0) = \lim_{h\to 0} \frac{f(x_0 + h\mathbf{u}) - f(x_0)}{h}$$

For a function f with three variables, the **gradient vector**, denoted by ∇f or **grad** f is

$$\nabla f =< f_x, f_y, f_z >= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Then, directional derivatives could be rewritten as

$$D_{\mathbf{u}}f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u}$$

Theorem 10.6.5: Maximizing the Directional Derivative

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(x)$ is $|\nabla f(x)|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(x)$.

Definition 10.6.6: Tangent Planes to Level Surfaces

The gradient vector at P, $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P.

If $\nabla F(x_0, y_0, z_0) \neq 0$, it is therefore natural to define the **tangent plane to the level surface** $F(x,y,z) = k$ **at** $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. We can write the equation of this tangent plane as

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane. Its symmetric equations are

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

10.7 Maximum and Minimum Values

Definition 10.7.1: Local Maximum/Minimum

A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) ins some disk with center (a, b)]. The number $f(a, b)$ is called **local maximum value**. If $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b), then $f(a, b)$ is **local minimum value**.
If the inequalities above hold for *all* points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

Theorem 10.7.2: Critical Points

If f has a local maximum or minimum at point (a, b) and the first-order derivatives of f exit there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.
A point is called **critical point** (or **stationary point**) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

Theorem 10.7.3: Second Derivative Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(1) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(2) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(3) If $D < 0$, then $f(a, b)$ is a **saddle point**.

(4) If $D = 0$, then the test is inconclusive.

Theorem 10.7.4: Extreme Value Theorem for Functions of Two Variables (EVT)

If f is **continous** on c **closed, bounded** set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Theorem 10.7.5: Absolute Maximum and Minimum Values

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

(1) Find the values of f at the critical points of f in D.

(2) Find the extreme values of f on the boundary of D.

(3) The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

10.8 Lagrange Multipliers

Definition 10.8.1: Lagrange Multiplier

In order to maximize or minimize a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, , y, z) = k$, we have **Lagrange Multiplier**. Suppose f has such an extreme value at a point $P(x_0, y_0, z_0)$. The gradient vectors $\nabla(x_0, y_0, z_0)$ and $\nabla(x_0, y_0, z_0)$ must be parallel. Therefore if $\nabla(x_0, y_0, z_0) \neq 0$, there is a number λ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

The number λ is called a **Lagrange Multiplier**.

Theorem 10.8.2: Method of Lagrange Multiplier

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$]:

(1) Find all values of x, y, z and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(2) Evaluate f at all the points (x,y,z) that results from step (1). The largest of these values is the maximum value of f ; the smallest is the minimum value of f.

Definition 10.8.3: Lagrange Multiplier for Two Constraints

We want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$. Suppose f has such an extreme value at a point $P(x_0, y_0, z_0)$. So there are numbers λ and μ (called **Lagrange Multiplier**) such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu h(x_0, y_0, z_0)$$

Chapter 11

Multiple Integrals

11.1 Double Integrals Over Rectangles

Definition 11.1.1: Riemann Sum

If $f(x)$ is defined for $a \leqslant x \leqslant b$, we start by dividing the interval $[a, b]$ into n subintervals $[x_i - x_{i-1}]$ and we choose sample points x_i^* in the subintervals. Then we form the Riemann sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

Definition 11.1.2: Double Integral of f over the rectangle R

The **double integral** of f over the rectangle R is

$$\iint_R f(x, y)\ \mathrm{d}A = \lim_{max \Delta x_i, \Delta y_i \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$

if this limit exists.

Property 11.1.3

If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\ \mathrm{d}A$$

Definition 11.1.4: Midpoint rule for double integrals

$$\iint_R f(x, y)\ \mathrm{d}A \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j})\Delta A$$

where $\overline{x_i}$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y_i}$ is the midpoint of $[y_{j-1}, y_j]$.

Theorem 11.1.5: Fubini's Theorem

If f is continuous on the rectangle $R = \{(x,y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then

$$\iint_R f(x,y)\, \mathrm{d}A = \int_a^b \int_c^d f(x,y)\, \mathrm{d}y\, \mathrm{d}x$$

Proposition 11.1.6

$$\iint_R g(x)h(y)\, \mathrm{d}A = \int_a^b g(x)\, \mathrm{d}x \int_c^d h(y)\, \mathrm{d}y \qquad \text{where} R = [a,b] \times [c,d]$$

11.2 Double Integrals Over General Regions

Definition 11.2.1: Double Integral of f over D

We define a new function F with domain R by angular region R.

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

If the double integral of F exists over R, then we define the **double integral of f over D** by

$$\iint_D f(x,y)\, \mathrm{d}A \qquad \text{where } F \text{ is given above}$$

Definition 11.2.2: Type I

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x,y) \mid a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x)\}$$

If f is continuous on a type I region D such that

$$D = \{(x,y) \mid a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x)\}$$

then

$$\iint_D f(x,y)\, \mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, \mathrm{d}y\, \mathrm{d}x$$

Definition 11.2.3: Type II

A plane region D is said to be of **type II** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x,y) \mid a \leqslant x \leqslant b, h_1(y) \leqslant y \leqslant h_2(y)\}$$

If f is continuous on a type I region D such that

$$D = \{(x, y) \mid c \leqslant y \leqslant d, h_1(y) \leqslant x \leqslant h_2(y)\}$$

then

$$\iint_D f(x, y)\, \mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

Property 11.2.4

$$\iint_D f(x, y)\, \mathrm{d}A = \iint_{D_1} f(x, y)\, \mathrm{d}A + \iint_{D_2} f(x, y)\, \mathrm{d}A$$

where $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries.

$$\iint_D 1\, \mathrm{d}A = A(D)$$

Theorem 11.2.5

If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$mA(D) \leqslant \iint_D f(x, y)\, \mathrm{d}A \leqslant MA(D)$$

11.3 Double Integrals In Polar Coordinates

Definition 11.3.1: Polar Rectangle

The polar coordinates (r, θ) of a point are related to rectangular coordinates (x, y) by the equations

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

A **polar rectangle** is

$$R = \{(r, \theta) \mid a \leqslant r \leqslant b, \alpha \leqslant \theta \leqslant \beta\}$$

Theorem 11.3.2: Change to Polar Coordinates in a Double Integral

If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b, \alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x, y)\, \mathrm{d}A = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r\, \mathrm{d}r\, \mathrm{d}\theta$$

Be careful not to forget the additional factor r on the right side of Formula 2.

Theorem 11.3.3

If f is continuous on a polar region of the form

$$D = \{(r,\theta) \mid \alpha \leqslant \beta, h_1(\theta) \leqslant r h_2(\theta)\}$$

then

$$\iint_D f(x,y)\,\mathrm{d}A = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta$$

11.4 Applications of Double Integrals

Property 11.4.1: Density and Mass

We obtain the total mass m of the lamina as the limiting value of approximations:

$$m = \lim_{\max \Delta x_i, \Delta y_j \to 0} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*) \Delta A_{ij} = \iint_D \rho(x,y)\,\mathrm{d}A$$

Definition 11.4.2: Moment

The **moment** of the entire lamina **about the x-axis**:

$$M_x = \lim_{\max \Delta x_i, \Delta y_j \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*) \Delta A_{ij} = \iint_D y\rho(x,y)\,\mathrm{d}A$$

The **moment about the y-axis**:

$$M_y = \lim_{\max \Delta x_i, \Delta y_j \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*) \Delta A_{ij} = \iint_D x\rho(x,y)\,\mathrm{d}A$$

Definition 11.4.3: Center of Mass

The coordinates $(\overline{x}, \overline{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x,y)$ are

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\rho(x,y)\,\mathrm{d}A \qquad \overline{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y\rho(x,y)\,\mathrm{d}A$$

where the mass m is given by

$$m = \iint_D \rho(x,y)\,\mathrm{d}A$$

Definition 11.4.4: Moment of Inertia

The **moment of inertia** (also called the **second moment**) of a particle of mass m about axis is defined to be mr^2, where r is the distance from the particle to the axis.

We divide D into small rectangles, approximate the moment of inertia of each sub rectangle about the x-axis, and take the limit of the sum as the sub rectangles become smaller. The result is the **moment of the inertia** of the lamina **about the x-axis**:

$$I_x = \lim_{\max_{\Delta x_i, \Delta y_j} \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) = \iint_D y^2 \rho(x, y) \, \mathrm{d}A$$

The **moment of the inertia about the y-axis**:

$$I_y = \lim_{\max_{\Delta x_i, \Delta y_j} \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) = \iint_D x^2 \rho(x, y) \, \mathrm{d}A$$

It is also of interest to consider the **moment of inertia about the origin**, also called the **polar moment of inertia**:

$$\begin{aligned} I_0 &= \lim_{\max_{\Delta x_i, \Delta y_j} \to 0} \sum_{i=1}^{m} [\sum_{j=1}^{n} (x_{ij}^*)^2 + (y_{ij}^*)] \rho(x_{ij}^*, y_{ij}^*) \Delta A_{ij} \\ &= \iint_D (x^2 + y^2) \rho(x, y) \, \mathrm{d}A \end{aligned}$$

11.5 Triple Integrals

Definition 11.5.1: Triple Integral

The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z) \, \mathrm{d}V = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{,} \sum_{k=1}^{n} f(x_i, y_k, z_k) \Delta V$$

Theorem 11.5.2: Fubini's Theorem for Triple Integrals

If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z) \, \mathrm{d}V = \int_r^s \int_c^d \int_a^b f(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

Definition 11.5.3: Triple Integral Over a General Bounded Region E

We define the **triple integral over a general bounded region E** in three dimensional space. We enclose E in a box B. Then we define a function F so that it agrees with f on E but is 0 for points in B that are outside E. By definition,

$$\iiint_E f(x, y, z) \, \mathrm{d}V = \iiint_B F(x, y, z) \, \mathrm{d}V$$

Definition 11.5.4: Type 1

A solid region E is said to be **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \leqslant z \leqslant u_2(x,y)\}$$

It can be shown that if E is a type 1 region, then

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,\mathrm{d}z\right] \mathrm{d}A$$

In particular, if the projection D of E onto the xy-plane is a type I plane region, then

$$E = \{(x,y,z) \mid a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x), u_1(x,y) \leqslant z \leqslant u_2(x,y)\}$$

and the equation becomes

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

If, on the other hand, D is a type II plane region, then

$$E = \{(x,y,z) \mid c \leqslant y \leqslant d, h_1(y) \leqslant x \leqslant h_2(y), u_1(x,y) \leqslant z \leqslant u_2(x,y)\}$$

and the equation becomes

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

Definition 11.5.5: Type 2

A solid region of **type 2** if it is of the form

$$E = \{(x,y,z) \mid (y,z) \in D, u_1(y,z) \leqslant x \leqslant u_2(y,z)\}$$

where, this time, D is the projection of E onto the yz-plane. The back surface is $x = u_1(y,z)$, the front surface is $x = u_2(y,z)$, and we have

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,\mathrm{d}x\right] \mathrm{d}A$$

Definition 11.5.6: Type 3

A solid region of **type 3** if it is of the form

$$E = \{(x,y,z) \mid (x,z) \in D, u_1(x,z) \leqslant y \leqslant u_2(x,z)\}$$

where, this time, D is the projection of E onto the yz-plane. The back surface is $x = u_1(y,z)$, the front surface is $x = u_2(y,z)$, and we have

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,\mathrm{d}y\right] \mathrm{d}A$$

Property 11.5.7: Application of Triple Integrals

The special case where $f(x,y,z) = 1$ for all points in E. Then all the triple integral does represent the volume of E :

$$V(E) = \iiint_E \mathrm{d}V$$

11.6 Triple Integral Coordinates

Definition 11.6.1: Cylinder Coordinates

To convert from cylindrical to rectangular coordinates, we use the equation

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

wheres to convert from rectangular to cylindrical coordinates, we use

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

Theorem 11.6.2: Formula for Triple Integration in Cylindrical Coordinates

This formula says that we convert a triple integral rectangular to cylindrical coordinates by writing $x = r\cos\theta, y = r\sin\theta$, leaving z as it is, using the appropriate limits of integration for z, r and θ, and replacing dV by $rdzdrd\theta$.

$$\iiint_E f(x,y,z)\,\mathrm{d}V = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

11.7 Triple Integrals in Spherical Coordinates

Definition 11.7.1: Spherical Coordinates

The **spherical coordinates** $(\rho, \theta, \phi$ of a point P in space, where $\rho = |OP|$ is the distance from the origin to P,θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP. Note that

$$\rho \leqslant 0 \qquad 0 \leqslant \phi \leqslant \pi$$

We use the equation below to convert from rectangular to spherical coordinates

$$x = \rho \sin \phi \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi$$

Also, the formula shows that

$$\rho^2 = x^2 + y^2 + z^2$$

Definition 11.7.2: Spherical Wedge

In the spherical coordinate system the counterpart of a rectangular box is a **spherical wedge**

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b, \alpha \leqslant \theta \leqslant \beta, c \leqslant \phi \leqslant d\}$$

Theorem 11.7.3

$$\iiint_E f(x, y, z)\, \mathrm{d}V = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)\rho^2\, \mathrm{d}\rho\, \mathrm{d}\theta\, \mathrm{d}\phi$$

11.8 Change of Variables in Multiple Integrals

Definition 11.8.1: Transformation

We consider a change of variables that is given by a **transformation** T from the uv-plane to the xy-plane:

$$T(u, v) = (x, y)$$

where x and y are related to u and v by the equations

$$x = g(u, v) \qquad y = h(u, v)$$

or, as we sometimes write

$$x = x(u, v) \qquad y = y(u, v)$$

We usually assume that T is a $\boldsymbol{C^1}$ **transformation**, which means that g and h have continuous first-order partial derivatives.
If T is a one-to-one transformation, then it has an **inverse transformation** T^{-1} from the xy-plane to the uv-plane and it may be possible to solve the equation for u and v in terms of x and y :

$$u = G(x, y) \qquad v = H(x, y)$$

Definition 11.8.2: Jacobian

The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Theorem 11.8.3: Change of Variables in a Double Integral

Suppose that T is a C^1 transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x,y)\,\mathrm{d}A = \iint_S f\big(x(u,v), y(u,v)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\mathrm{d}v$$

Definition 11.8.4: Jacobian of Triple Integral

The **Jacobian** of T is a 3×3 determinant. We have the following formula for triple integrals:

$$\begin{aligned}&\iiint_R f(x,y,z)\,\mathrm{d}V \\ &= \iiint_S f\big(f(u,v,w), y(u,v,w), z(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \mathrm{d}u\mathrm{d}v\mathrm{d}w\end{aligned}$$

Chapter 12

Vector Calculus

12.1 Vector Fields

Definition 12.1.1: Vector Field on $\mathbb{R}^2$

Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function **F** that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

Definition 12.1.2: Vector Field on $\mathbb{R}^3$

Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function **F** that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

12.2 Line Integrals

Definition 12.2.1: Line Integral of f along C

If f is defined on a smooth curve C given by

$$x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

then the **line integral of f along C** is

$$\int_C f(x, y)\,\mathrm{d}s = \lim_{max\Delta s_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

Property 12.2.2: Use Length of C to evaluate

$$\int_C f(x, y)\,\mathrm{d}s = \int_a^b f(x(t), y(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,\mathrm{d}t$$

Definition 12.2.3: Line Integral with respect to x and y

$$\int_C f(x,y)\,\mathrm{d}x = \int_a^b f(x(t),y(t))x'(t)\,\mathrm{d}t$$
$$\int_C f(x,y)\,\mathrm{d}y = \int_a^b f(x(t),y(t))y'(t)\,\mathrm{d}t$$

Definition 12.2.4: Line Integral of Vector Fields

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,\mathrm{d}s = \int_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s$$

Property 12.2.5

$$\int_C \mathbf{F}\cdot\,\mathrm{d}r = \int_C P\,\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z \qquad \text{where } \mathbf{F} = P\mathbf{i} + Q\mathbf{k} + R\mathbf{k}$$

12.3 The Fundamental Theorem for Line Integrals

Theorem 12.3.1: The Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,\mathrm{d}x = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Theorem 12.3.2: The Fundamental Theorem for Line Integrals

Let C be a smooth curve given by the vector function $\mathbf{r}(t), a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f\cdot\,\mathrm{d}r = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Theorem 12.3.3: Independent of Path

$\int_C F\cdot\,\mathrm{d}r$ is independent of path in D if and only if $\int_C F\cdot\,\mathrm{d}\mathbf{r} = 0$ for every closed path C in D.

Theorem 12.3.4

Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector filed on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Theorem 12.3.5

If $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Theorem 12.3.6: Test Conservative

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

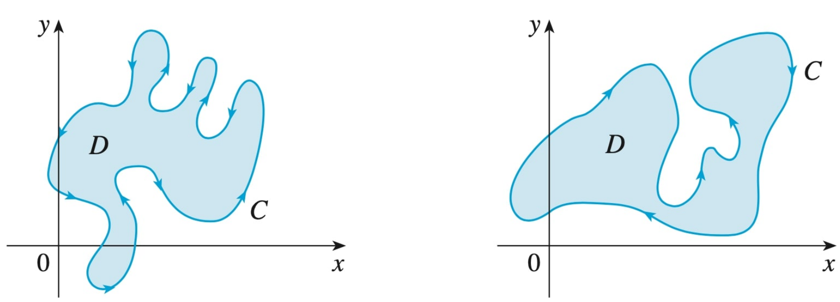
Then $\mathbf{F}$ is conservative.

12.4 Green's Theorem

Theorem 12.4.1: Green's Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \left(\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A$$



(a) Positive orientation (b) Negative orientation

Property 12.4.2

The Green's Theorem gives the following formulas for the area of D:

$$A = \oint_C x\,\mathrm{d}y = -\oint_C y\,\mathrm{d}x = \frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x$$

12.5 Curl and Divergence

Definition 12.5.1: Curl

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivative of P, Q, and R all exist, then the **curl** of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Remember the definition by means of the symbolic expression:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$$

Theorem 12.5.2

If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\nabla \mathbf{f}) = \mathbf{0}$$

Theorem 12.5.3

If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose components functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

Definition 12.5.4: Divergence of F

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector on $\mathbb{R}^3$ and $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, and \frac{\partial R}{\partial z}$ exist, then the **divergence of F** is the function of three variables defined by

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

In terms of the gradient operator, the divergence of $\mathbf{F}$ can be written symbolically:

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

Theorem 12.5.5

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

Theorem 12.5.6: Vector Forms of Green's Theorem

$$\oint_C \mathbf{F}\, \mathrm{d}\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, \mathrm{d}A$$

A second vector form of Green's Theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}t = \iint_D \operatorname{div} \mathbf{F}(x, y)\, \mathrm{d}A$$

12.6 Parametric Surface and Their Areas

Definition 12.6.1: Parametric Surface

We suppose that

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

is a vector-valued function defined on a region D in the uv-plane. So x, y, and z, the component functions of r , are functions of the two variables u and v with domain D. The set of all points (x, y, z) in $\mathbb{R}^3$ such that

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

and (u, v) varies throughout D, is called **parametric surface** S and the second equation is called **parametric equations of** S.

Definition 12.6.2: Parametric Surface

If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, \mathrm{d}A$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

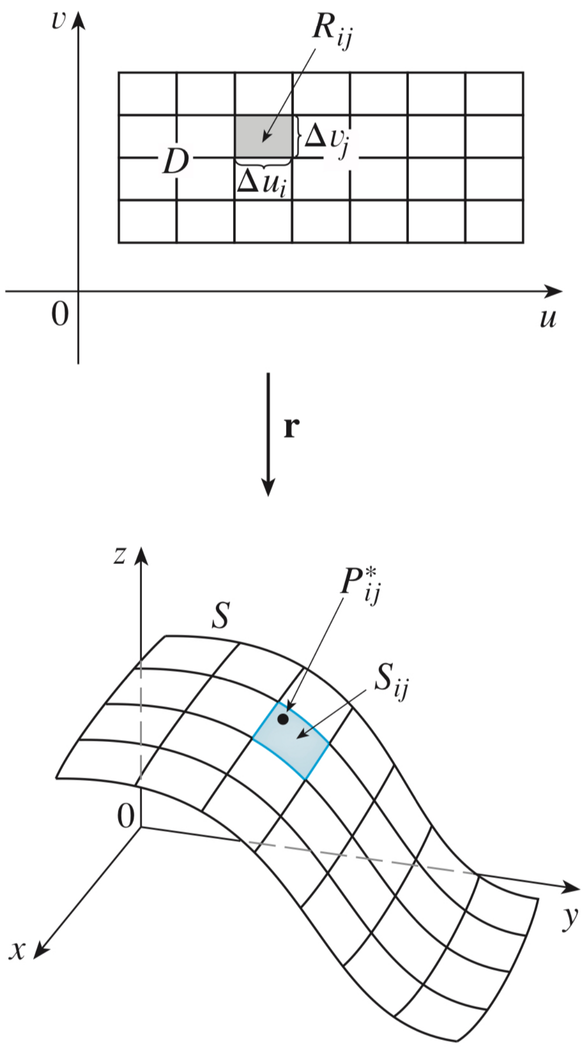
Definition 12.6.3: Surface Area Formula

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, \mathrm{d}A$$

12.7 Surface Integrals

Definition 12.7.1: Surface Integral of f over the surface S

$$\iint_S f(x,y,z)\,\mathrm{d}S = \lim_{\max \Delta u_i, \Delta v_j \to 0} \sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$$



If the components are continuous, and $\mathbf{r}_u, \mathbf{r}_v$ are nonzero and nonparallel in the interior of D, it can be shown from Definition 1, even when D is not a rectangle, that

$$\iint_S f(x,y,z)\,\mathrm{d}S = \iint_S f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}A$$

Observe also that

$$\iint_S 1\,\mathrm{d}S = \iint_S |\mathbf{r}_u \times \mathbf{r}_v|\,\mathrm{d}A$$

Definition 12.7.2: Surface Integrals in Graph point of view

An surface S with the equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equation

$$x = x \qquad y = y \qquad z = g(x, y)$$

and so we have

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right)\mathbf{k} \qquad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right)\mathbf{k}$$

Thus

$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Therefore, in this case, Formula 2 becomes

$$\iint_S f(x, y, z)\, \mathrm{d}S = \iint_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, \mathrm{d}A$$

Definition 12.7.3: Surface Integral in Oriented Surfaces

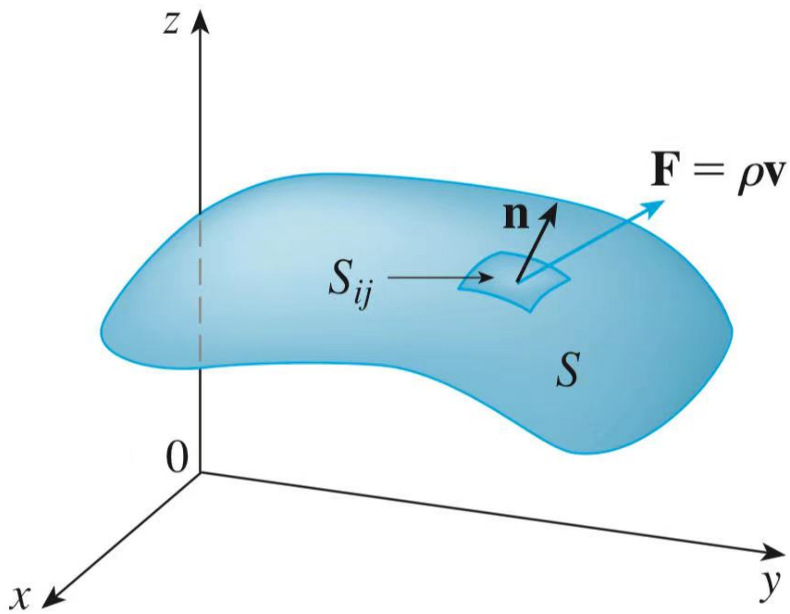
For a surface $z = g(x, y)$ given as the graph of g, we use Equation 3 to associate with the surface a natural orientation given by the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} = \frac{-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}}$$

Definition 12.7.4: Surface Integral of Vector Fields

If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit vector $\mathbf{n}$, then the **surface integral of F over S** is

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbb{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, \mathrm{d}S$$



This integral is also called the **flux** of **F** across S.

If S is given by a vector function $\mathbf{r}(u, v)$, then n is given by Equation 6, we have

$$\iint_S \mathbf{F} \cdot \mathrm{d}S = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, \mathrm{d}A$$

In the case of a surface S given by a graph $z = g(x, y)$, we can think of x and y as parameters and use Equation 3 to write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} \right)$$

Thus **Surface integrals of vector fields in graph point of view** is

$$\iint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \iint_S \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) \mathrm{d}A$$

12.8 Stoke's Theorem

Theorem 12.8.1: Stoke's Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbb{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S \mathrm{curl}\mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

12.9 The Divergence Theorem

Theorem 12.9.1: The Divergence Theorem

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} = \iiint_E \mathrm{div}\mathbf{F} \, \mathrm{d}V$$